Symmetrization of the Berezin Star Product and Multiple Star Product Method

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Abstract

We construct a multiple star product method and by using this method, show that integral forms of some star products can be written in terms of the path-integral. This method can be applied to some examples. Especially, the associativity of the skew-symmetrized Berezin star product proposed in [SW], is recovered in large N limit of the multiple star product. We also derive the path integral form of the Kontsevich star product from the multiple Moyal star product. This paper includes some reviews about star products.

1 Introduction

In recent years, a relation between superstring theory and a deformation quantization has been explored. D-branes, boundaries of open strings, are non-perturbative object of superstring theory. Matrix Models [BFSS, IKKT] were proposed as a D-brane action a few years ago. It is shown that a stable solution of this action is non-commutative manifold [CDS, AIIKKT]. This non-commutativity, however, comes from the Moyal quantization. So, this solution implies a flat D-brane‡. If we regard some D-brane as space-time, we should study the deformation quantization of curved spaces in order to realize the quantum gravity.

In order to proceed further it will be useful to clarify mathematical background of the deformation quantization. The star product was first introduced by Groenewold[Gr], which is now known as the Moyal product[Mo]. They associate an operator product to a noncommutative product of functions. Here, the operators are mapped into the functions

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‡ We also derive a noncommutative gauge theory on a fuzzy sphere from the matrix model[IKTW]

by taking account the Weyl ordering. The Weyl ordering means a skew-symmetric definition as we will see later. Also Berezin tried to quantize curved phase spaces about 25 years ago and succeeded to quantize some Kähler manifold e.g. sphere[Be]. Recently the Berezin quantization has been generalized to arbitrary Kähler manifold[RT]. However the Berezin quantization is defined without skew symmetry, hence is not a generalization of Moyal one.

The correlation between these methods of quantization is, however, not clear at all. In this paper, we attempt to skew-symmetrize the Berezin quantization by means of the multiple star product method. The multiple star products reduce to the path-integral form in large N limit. (See [Sh, Al] for the original ideas.) As a result, our formulation turns similar to the path-integral form of the Kontsevich quantization which is defined perturbatively on Poisson manifold[Ko] but also can be described by a bosonic string path-integral[CF]. Especially in the flat case, our star product coincides with the Kontsevich star product.

This paper consists of the following sections. In section 2, we review the deformation quantization e.g. Moyal [Mo], Berezin [Be, MM] and Kontsevich [Ko, CF] quantization. In section 3, we first construct the multiple star product method and explain the symmetrized Berezin (or Wick type) star product[SW, Ma]. We also study its associativity in detail. In section 4, we derive the path integral form of the Kontsevich star product on the flat plane from the multiple star product method. Section 5 is devoted to discussions. Appendix includes some examples that the multiple star product method is available.

2 Deformation Quantization

This section includes the definition and properties of the deformation quantization. We also review Moyal, Berezin and Kontsevich quantization briefly as examples.

2.1 General Definition and Property

The deformation quantization[BFFLS, St] is provided by a star product, which is defined by

$$f * g = \sum_{m=0}^{\infty} B_m(f,g)\lambda^m, \tag{1}$$

where

- λ is a deformation parameter,
- $f, g \in A = C^\infty(M)[\![\lambda]\!]$
  $C^\infty(M)[\![\lambda]\!]$ means that the coefficients of λ power series are C^∞ functions on M ,
- B_m are bi-differential operators ($B_m : A \times A \to A$),

The deformation quantization has the following properties:

1. associativity
$$f * (g * h) = (f * g) * h. \tag{2}$$

2. $m = 0$
$$B_0(f, g) = fg. \tag{3}$$

3. $m = 1$
$$B_1(f, g) - B_1(g, f) = \{f, g\} = 2 \sum_{i,j} \alpha^{ij} \partial_i f \partial_j g, \tag{4}$$

where $i, j = 1, 2, \cdots, d = dim(M)$ and $\{\cdot, \cdot\}$ is a Poisson bracket which satisfies

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h\{f, g\}\} = 0, \tag{5}$$

so the skew-symmetric bivector field α satisfies

$$\alpha^{il} \partial_l \alpha^{jk} + \alpha^{jl} \partial_l \alpha^{ki} + \alpha^{kl} \partial_l \alpha^{ij} = 0. \tag{6}$$

It can be shown that one or more star product determined by (2)(3)(4) exist. The deformation quantization has the following equivalence called a gauge equivalence. $*$ and $*'$ are identified if
$$f' *' g' = D(f * g), \tag{7}$$
where $f' = D(f)$, $g' = D(g)$ and D is a differential operator ($D : A \to A$). However, we can take two simple gauges. One is the skew-symmetric gauge

$$B_1(f, g) = \frac{1}{2}\{f, g\} = \sum_{i,j} \alpha^{ij} \partial_i f \partial_j g. \tag{8}$$

The other gauge is
$$B_1(f, g) = \sum_{i,j} \beta^{ij} \partial_i f \partial_j g, \tag{9}$$

where β is the upper triangle matrix of α which satisfies $\alpha^{ij} = \beta^{ij} - \beta^{ji}$. So we call the star product determined by (8) and (9) the skew-symmetric and asymmetric product respectively. We have three concrete examples of deformation quantization, which are put together in the following table.

Manifold	flat plane	Kähler	Poisson
Quantization	Moyal	Berezin	Kontsevich
Symbol	$\star$	$\boxed{\star}$	$*$
$m = 1$	skew-symmetric	asymmetric	skew-symmetric

We consider real two dimensional manifolds for the sake of simplicity from now on.

2.2 Moyal Quantization

The Poisson bracket on the flat plane is defined by

$$\{f,g\} = \sum_{i,j} \varepsilon^{ij} \partial_i f \partial_j g = \partial_x f \partial_p g - \partial_p f \partial_x g. \tag{10}$$

Thus $\alpha^{ij} = \varepsilon^{ij}/2$ by (4). So we obtain the associative star product on the flat plane i.e. the Moyal star product as

$$\begin{aligned} f \star g(x,p) &= f(x,p) e^{\frac{\lambda}{2}(\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x})} g(x,p) \\ &= fg + \lambda \frac{1}{2}\{f,g\} + O(\lambda^2). \end{aligned} \tag{11}$$

This star product agrees with eq.(3) and (8), and satisfies the associativity from the following results

$$e_1 \star (e_2 \star e_3) = e^{-\frac{\lambda}{2}(m_1 n_2 + m_2 n_3 + n_3 m_1 - n_1 m_2 - n_2 m_3 - m_3 n_1)} e_1 e_2 e_3 = (e_1 \star e_2) \star e_3, \tag{12}$$

where e_i's are the Fourier series

$$e_i = e^{i(m_i x + n_i p)}.$$

Here we require a usual canonical commutation relation

$$[x,p]_\star = x \star p - p \star x = i\hbar, \tag{13}$$

so that we obtain $\lambda = i\hbar$. Also this star product can be written by the integral form[Ba], because we have the following relations

$$e^{i(mx+np)} \star e^{i(m'x+n'p)} = e^{-\frac{i\hbar}{2}(mn'-m'n)} e^{i(mx+np)} e^{i(m'x+n'p)} \tag{14}$$

and

$$\int \frac{dw d\eta}{\pi\hbar} \frac{dw' d\eta'}{\pi\hbar} e^{\frac{2i}{\hbar}S} e^{i(mw+n\eta)} e^{i(m'w'+n'\eta')} = e^{-\frac{i\hbar}{2}(mn'-m'n)} e^{i(mx+np)} e^{i(m'x+n'p)}, \tag{15}$$

where

$$S = \begin{vmatrix} 1 & 1 & 1 \\ x & w & w' \\ p & \eta & \eta' \end{vmatrix}.$$

The left hand sides of eq.(14) and (15) are equivalent, so that we obtain the integral form of the Moyal star product after multiplying arbitrary Fourier coefficients and integrating over m, n as

$$f \star g(x,p) = \int \frac{dw d\eta}{\pi\hbar} \frac{dw' d\eta'}{\pi\hbar} e^{\frac{2i}{\hbar}S} f(w,\eta) g(w',\eta'). \tag{16}$$

2.3 Berezin Quantization

The Poisson bracket on the Kähler manifold is given as

$$\{f,g\} = \frac{2}{i} h^{z\bar{z}} (\partial_z f \partial_{\bar{z}} g - \partial_{\bar{z}} f \partial_z g), \tag{17}$$

where $h^{z\bar{z}}$ is the inverse of a Kähler metric

$$h_{z\bar{z}} = \partial_z \partial_{\bar{z}} K(z,\bar{z}),$$

and $K(z,\bar{z})$ is a Kähler potential. The factor $2/i$ in eq.(17) is necessary in order that the Poisson bracket becomes eq.(10) in the flat case. The original Berezin quantization covers only Ricci flat Kähler manifold. The Berezin star product is defined by

$$f \boxtimes g(z,\bar{z}) = \int d\mu_\nu(v,\bar{v}) e^{\frac{1}{\nu}\Phi(z,\bar{z},v,\bar{v})} f(z,\bar{v}) g(v,\bar{z}), \tag{18}$$

where $\Phi(z,\bar{z},v,\bar{v})$ is called the Calabi function and defined by the Kähler potential as

$$\Phi(z,\bar{z},v,\bar{v}) = K(z,\bar{v}) + K(v,\bar{z}) - K(z,\bar{z}) - K(v,\bar{v}), \tag{19}$$

and the measure $d\mu_\nu$ is determined by the metric as

$$d\mu_\nu(z,\bar{z}) = h_{z\bar{z}} \frac{i dz \wedge d\bar{z}}{2\pi\nu}. \tag{20}$$

This star product can be expanded around $\lambda = \nu/2i = 0$ as follows

$$f \boxtimes g(z,\bar{z}) = fg + \lambda\Big(B_1^+(f,g) + B_1^-(f,g)\Big) + O(\lambda^2), \tag{21}$$

where B_1^+ and B_1^- are a symmetric part and a skew-symmetric part respectively as

$$B_1^+ = 2iAfg - \frac{1}{2}\{f,g\}_+ \ , \ B_1^- = \frac{1}{2}\{f,g\} \tag{22}$$

and

$$A = \frac{1}{2} h^{z\bar{z}} \partial_z \partial_{\bar{z}} \log h_{z\bar{z}} \ , \ \{f,g\}_+ = \frac{2}{i} h^{z\bar{z}} (\partial_z f \partial_{\bar{z}} g + \partial_{\bar{z}} f \partial_z g).$$

If the manifold M is the Kähler manifold, $A = 0$ [RT]. Thus (3) and (4) are satisfied. Also the associativity is shown as the following.

$$\Big((f \boxtimes g) \boxtimes h\Big)(z,\bar{z}) = \int d\mu_\nu(v,\bar{v}) d\mu_\nu(u,\bar{u}) f(z,\bar{v}) g(v,\bar{u}) h(u,\bar{z}) e^{\frac{1}{\nu}\left(\Phi(z,\bar{u},v,\bar{v}) + \Phi(z,\bar{z},u,\bar{u})\right)}.$$

$$\Big(f \boxtimes (g \boxtimes h)\Big)(z,\bar{z}) = \int d\mu_\nu(v,\bar{v}) d\mu_\nu(u,\bar{u}) f(z,\bar{v}) g(v,\bar{u}) h(u,\bar{z}) e^{\frac{1}{\nu}\left(\Phi(v,\bar{z},u,\bar{u}) + \Phi(z,\bar{z},v,\bar{v})\right)}.$$

The Calabi function clearly satisfies

$$\Phi(z,\bar{u},v,\bar{v}) + \Phi(z,\bar{z},u,\bar{u}) = \Phi(v,\bar{z},u,\bar{u}) + \Phi(z,\bar{z},v,\bar{v}),$$

so the associativity is shown:

$$\Big((f \boxtimes g) \boxtimes h\Big)(z,\bar{z}) = \Big(f \boxtimes (g \boxtimes h)\Big)(z,\bar{z}). \tag{23}$$

2.4 Kontsevich Quantization

The Kontsevich quantization covers the Poisson manifold (M) which is a general manifold with the Poisson structure. He perturbatively solved $B_m(f,g)$'s under the conditions (2),(3) and (8) as

$$B_m(f,g) = \sum_{\Gamma \in G_m} w_\Gamma B_\Gamma(f,g), \tag{24}$$

where G_m is a set of diagrams related to the number m , $B_\Gamma(f,g)$ is a bi-differential operator determined by the Feynman diagram and ω_Γ is a weight [Ko]. Thus Kontsevich defines the star product on the Poisson manifold by a formal power series of λ as

$$f * g = \sum_{m=0}^{\infty} \lambda^m \sum_{\Gamma \in G_m} w_\Gamma B_\Gamma(f,g). \tag{25}$$

Also Cattaneo and Felder have shown that the Kontsevich star product (25) coincides with the path integral form of a topological bosonic string (non-linear sigma model):

$$f * g(x) = \int_{X(\infty)=x} f(X(1)g(X(0))e^{\frac{i}{\hbar}S[X,\eta]}\mathcal{D}X\mathcal{D}\eta, \tag{26}$$

where the action is defined on a disk D as

$$S[X,\eta] = \int_D \eta_i(u) \wedge dX^i(u) + \frac{1}{2}\alpha^{ij}(X(u))\eta_i(u) \wedge \eta_j(u),$$

and

- $D = \{u \in R^2 \ , \ |u| \leq 1\}$,
- X and η are real bosonic fields,
- $X : D \to M$,
- η is a differential 1-form on D : $X^*(T^*M) \otimes T^*D$.

In the symplectic case, the action can be integrated over η and becomes a boundary integration by the Stokes's theorem as

$$f *_{\mathrm{symp}} g(x) = \int_{\gamma(\pm\infty)=x} f(\gamma(1))g(\gamma(0))e^{\frac{i}{\hbar}\int_\gamma d^{-1}\omega}d\gamma, \tag{27}$$

where γ is a loop trajectory from x to x .

3 Symmetrized Berezin star product

In this section, we first explain the multiple star product method. Next we define the S-star product to clarify the relationship between Moyal and Berezin quantization. However the S-star product is not associative in the curved space. So using the multiple star product method, we derive an associative star product i.e. the O-star product.

3.1 Multiple star product method

Generally the integral forms of the star products can be written as

$$f \diamond g(\alpha) = \int d\mu_\lambda(\beta,\gamma)\ e^{\mathcal{K}_\lambda(\alpha,\beta,\gamma)} f(\beta) g(\gamma), \tag{28}$$

where $\alpha, \beta, \gamma \in M$, $\mathcal{K}_\lambda = \mathcal{K}/\lambda$ is an integral kernel and $d\mu_\lambda = d\mu/\lambda^2$ is a measure which relates two points on M. We assume that this star product $\diamond$ satisfies the followings:

$$f \diamond g = fg + \lambda \frac{\{f,g\}}{2} + O(\lambda^2), \tag{29}$$

$$f \diamond 1 = 1 \diamond f = f, \tag{30}$$

$$d\mu_\lambda(\beta,\gamma) = d\mu_\lambda(\gamma,\beta). \tag{31}$$

We also add a assumption $\mathcal{K}_\lambda(\alpha,\beta,\beta) = 0$ in particular. Note that we don't require this star product $\diamond$ is associative. Then we call this product $\diamond$ the *non-associative* star product.

Next we define the multiple star product of $\diamond$ as

$$A^N(f) = f_{N/N} \diamond f_{N-1/N} \diamond \cdots \diamond f_{2/N} \diamond f_{1/N}. \tag{32}$$

An equivalence of the forward product $\overleftarrow{A}^N$ and the backward product $\overrightarrow{A}^N$ is necessary at least in order that A^N is well-defined where

$$\begin{aligned} \overleftarrow{A}^N(f) &:= (f_{N/N} \diamond (f_{N-1/N} \diamond (\cdots \diamond (f_{1/N} \diamond 1)\cdots))) \\ &= \int \Big(\prod_{i=1}^N d\mu_{\lambda N}(\beta_{i/N},\gamma_{i/N}) f_{i/N}(\beta_{i/N}) \Big) \exp \sum_{i=1}^N \frac{1}{N} \mathcal{K}_\lambda(\gamma_{i+1/N},\beta_{i/N},\gamma_{i/N}), \end{aligned} \tag{33}$$

$$\begin{aligned} \overrightarrow{A}^N(f) &:= (((\cdots (1 \diamond f_{N/N}) \diamond \cdots) \diamond f_{2/N}) \diamond f_{1/N}) \\ &= \int \Big(\prod_{i=1}^N d\mu_{\lambda N}(\beta_{i/N},\gamma_{i/N}) f_{i/N}(\beta_{i/N}) \Big) \exp \sum_{i=1}^N \frac{1}{N} \mathcal{K}_\lambda(\gamma_{i-1/N},\gamma_{i/N},\beta_{i/N}), \end{aligned} \tag{34}$$

$$\alpha = \beta_0 = \beta_{N+1/N} = \gamma_0 = \gamma_{N+1/N}. \tag{35}$$

Note that we change the deformation parameter λ to λN. ¿From this equivalence, we obtain a condition

$$\frac{1}{N} \sum_{i=1}^N \Big(\mathcal{K}_\lambda(\gamma_{i+1/N},\beta_{i/N},\gamma_{i/N}) - \mathcal{K}_\lambda(\gamma_{i-1/N},\gamma_{i/N},\beta_{i/N}) \Big) = 0. \tag{36}$$

Using the boundary condition (35) and the additional condition $\mathcal{K}(\alpha,\beta,\beta) = 0$, this condition (36) is also deformed as

$$\frac{1}{N} \sum_{i=0}^N \Big(\mathcal{K}_\lambda(\gamma_{i+1/N},\beta_{i/N},\gamma_{i/N}) - \mathcal{K}_\lambda(\gamma_{i/N},\gamma_{i+1/N},\beta_{i+1/N}) \Big) = 0. \tag{37}$$

This condition corresponds to the associativity condition in the case of $f_{i/N} = 1$ except for three $f_{i/N}$'s. We denote $A^N(f) := \overleftarrow{A}^N(f) = \overrightarrow{A}^N(f)$ when $\mathcal{K}_\lambda$ satisfies eq.(37).

3.2 Relationship between Moyal and Berezin Star Product

The Berezin star product in the flat case, coincides with the Moyal's one except for skew-symmetry or asymmetry. This difference is explained as follows. First, we write the Moyal star product in complex variables to make clear the correspondence to Berezin star product,

$$f \star g(z, \bar{z}) = f(z, \bar{z}) e^{\hbar(\overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}} - \overleftarrow{\partial_{\bar{z}}}\overrightarrow{\partial_z})} g(z, \bar{z}), \tag{38}$$

where $z = x + ip$. This star product is gauge equivalent(7) to $\star_{st}$ and $\star_{ar}$ where

$$\star_{st} = e^{2\hbar\overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}}} \quad \text{and} \quad \star_{ar} = e^{-2\overleftarrow{\partial_{\bar{z}}}\overrightarrow{\partial_z}}, \tag{39}$$

because the gauge equivalent condition is satisfied in the case of

$$D = e^{\hbar\overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}}} \quad \text{and} \quad D = e^{-\hbar\overleftarrow{\partial_{\bar{z}}}\overrightarrow{\partial_z}}, \tag{40}$$

respectively[Vo, Be, APS]. Thus we obtain a star product relation,

$$\star = (\star_{st}\star_{ar})^{\frac{1}{2}}. \tag{41}$$

Here $\star_{st}^{\frac{1}{2}}$ and $\star_{ar}^{\frac{1}{2}}$ can be written in the integral forms[APS] as

$$\begin{aligned} f \star_{st}^{\frac{1}{2}} g(z, \bar{z}) &= \int \frac{idw \wedge d\bar{w}}{2\pi\theta} e^{\frac{1}{\theta}|w-z|^2} f(w, \bar{z}) g(z, \bar{w}), \\ f \star_{ar}^{\frac{1}{2}} g(z, \bar{z}) &= \int \frac{idv \wedge d\bar{v}}{2\pi(-\theta)} e^{-\frac{1}{\theta}|v-z|^2} f(z, \bar{v}) g(v, \bar{z}), \end{aligned} \tag{42}$$

where $\theta = -\hbar$. Thus we obtain

$$f \star g(z, \bar{z}) = -\int \frac{idv \wedge d\bar{v}}{2\pi\theta} \frac{idw \wedge d\bar{w}}{2\pi\theta} e^{\frac{1}{\theta}(-|v-z|^2 + |w-z|^2)} f(w, \bar{v}) g(v, \bar{w}). \tag{43}$$

The star products (42) are two types of the Berezin star product on the flat plane i.e. the term $-|v-z|^2$ is the Calabi function on the flat plane. Taking this result into account, we generalize the Moyal star product $\star$ to the S-star product on the Ricci flat Kähler manifold [§], which is defined by

$$f \boxtimes g(z, \bar{z}) := \int d\mu_\theta(v, \bar{v}) d\mu_{-\theta}(w, \bar{w}) \exp \frac{1}{\theta}\Big(\Phi(z, \bar{z}; v, \bar{v}) - \Phi(z, \bar{z}; w, \bar{w})\Big) f(w, \bar{v}) g(v, \bar{w}), \tag{44}$$

where $d\mu_\theta(z, \bar{z}) = h_{z\bar{z}} idz \wedge d\bar{z}/2\pi\theta$ similarly to the definition (20). However, this star product is not associative unless flat. This complication is overcome by using the multiple star product method.

[§] In [Ma], it is discussed that the S-star product may be available to general Kähler manifold.

3.3 Associativity of Symmetrized Berezin Star Product

In this section, we attempt to recover the associativity of the S-star product. First, we show that the S-star product is non-associative star product. In the case of the S-star product, we know the following correspondence:

$$\alpha = (z, \bar{z}),\ \beta = (w, \bar{v}),\ \gamma = (v, \bar{w}), \tag{45}$$

$$\lambda = \theta,\ d\mu_\lambda(\beta, \gamma) = d\mu_\theta(v, \bar{v}) d\mu_{-\theta}(w, \bar{w}), \tag{46}$$

$$\mathcal{K}_\lambda(\alpha, \beta, \gamma) = \frac{1}{\theta}\Big(\Phi(z, \bar{z}; v, \bar{v}) - \Phi(z, \bar{z}; w, \bar{w})\Big). \tag{47}$$

Thus the S-star product clearly satisfies the conditions (31) and $\mathcal{K}_\lambda(\alpha, \beta, \beta) = 0$. Also it is derived in [Ma] that this product satisfies the conditions (29) and (30).

Next, we survey whether the S-star product satisfies the condition (37) or not. In the S-star product, *lhs* of (37) is written in terms of the Kähler potential K as

$$\begin{aligned} lhs \ &= \ \frac{1}{\theta}\frac{1}{N}\sum_{i=0}^{N}\Big(K(v_{i+1/N}, \bar{v}_{i/N}) + K(v_{i/N}, \bar{v}_{i+1/N}) - 2K(v_{i/N}, \bar{v}_{i/N})\Big) \\ &\quad -\Big(K(w_{i+1/N}, \bar{w}_{i/N}) + K(w_{i/N}, \bar{w}_{i+1/N}) - 2K(w_{i/N}, \bar{w}_{i/N})\Big). \end{aligned} \tag{48}$$

This result is non-zero but becomes zero in the large N limit as

$$lhs \to \frac{1}{\theta}\int_0^1 d\tau\ d\Big(K(v, \bar{v}) - K(w, \bar{w})\Big) = 0, \tag{49}$$

where $v, w = v(\tau), w(\tau)$ and the boundary conditions (35) become

$$v(0) = v(1) = w(0) = w(1) = z,\ \bar{v}(0) = \bar{v}(1) = \bar{w}(0) = \bar{w}(1) = \bar{z}. \tag{50}$$

Thus $A^N(f)$ is ill-defined but $A(f) = \lim_{N\to\infty} A^N(f)$ is well-defined. Then we call this star product *pseudo-associative* product. By using $A(f)$ and

$$f_{i/N} = \begin{cases} g & (i/N = i_1/N \to \tau_1) \\ f & (i/N = i_2/N \to \tau_2) \\ 1 & (i/N \to \tau \neq \tau_1, \tau_2) \end{cases}\ , \tag{51}$$

we construct an associative star product $\circledast$ in terms of the path integral form as

$$\begin{aligned} & f \circledast g(z, \bar{z}) := A(f) \\ = \ & \cdots 1 \boxtimes 1 \boxtimes f \boxtimes 1 \boxtimes 1 \cdots 1 \boxtimes 1 \boxtimes g \boxtimes 1 \boxtimes 1 \cdots \\ = \ & \lim_{N\to\infty} \int \prod_{i=1}^{N} d\mu_{\theta N}(v_{i/N}, \bar{v}_{i/N}) d\mu_{-\theta N}(w_{i/N}, \bar{w}_{i/N}) \exp\Big[\frac{i}{\theta}S_i\Big] f(v_{i_2/N}, \bar{w}_{i_2/N}) g(v_{i_1/N}, \bar{w}_{i_1/N}) \\ = \ & \int \mathcal{D}\mu_\theta(v, \bar{v}) \mathcal{D}\mu_{-\theta}(w, \bar{w}) \exp\Big[\frac{i}{\theta}S\Big] f\Big(v(\tau_2), \bar{w}(\tau_2)\Big) g\Big(v(\tau_1), \bar{w}(\tau_1)\Big), \end{aligned} \tag{52}$$

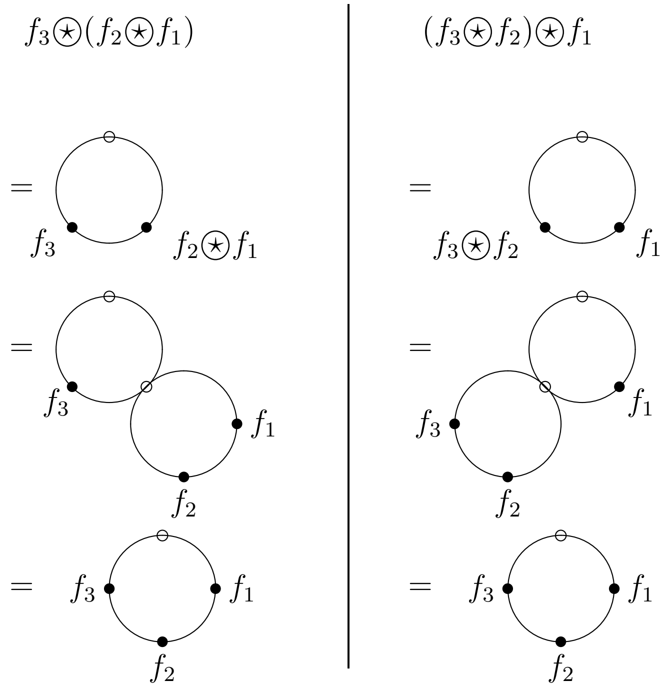

Figure 1: If we denote O-star product by a circle, the associativity is shown as above. Here $\circ$ point means the boundary of O-star product as $(z, \bar{z})$.

where actions are written as follows

$$S_i = \frac{1}{i}\frac{1}{N}\sum_{i=1}^{N}\Big(\Phi(v_{i+1/N}, \bar{w}_{i+1/N}; v_{i/N}, \bar{v}_{i/N}) - \Phi(v_{i+1/N}, \bar{w}_{i+1/N}; w_{i/N}, \bar{w}_{i/N})\Big), \tag{53}$$

$$S = \frac{1}{i}\int_{-\infty}^{\infty}\Big[\Big(\psi(v,\bar{v}) - \psi(v,\bar{w})\Big)\frac{\partial v}{\partial \tau} - \Big(\bar{\psi}(w,\bar{w}) - \bar{\psi}(v,\bar{w})\Big)\frac{\partial \bar{v}}{\partial \tau}\Big]d\tau, \tag{54}$$

and the path integral measure is defined by

$$\mathcal{D}\mu_\theta(v,\bar{v}) := \lim_{N\to\infty}\prod_{i=1}^{N} d\mu_{\theta N}(v_{i/N}, \bar{v}_{i/N}). \tag{55}$$

Note that $\psi(z,\bar{z})$ is a canonical conjugation of z, which is defined by

$$\psi(z,\bar{z}) := \frac{\partial K(z,\bar{z})}{\partial z}. \tag{56}$$

As above, associative symmetrized Berezin star product is defined as O-star product correctly. The associativity is satisfied as illustrated in Figure 1.

4 Construction of Kontsevich Star Product from Multiple Star Product Method

In this section, we show that the multiple Moyal star product corresponds to the path integral form of the Kontsevich star product on the flat plane. First preparatory to this derivation, we write the multiple Moyal star product in large N limit as

$$\begin{aligned} A_\star(f) &:= \lim_{N\to\infty} f_{N/N}(x,p)\star f_{N-1/N}(x,p)\star\cdots\star f_{2/N}(x,p)\star f_{1/N}(x,p) \\ &= \lim_{N\to\infty}\int\prod_{i=1}^{N}\frac{d\xi_{i/N}d\eta_{i/N}}{\pi\hbar N}\frac{d\xi'_{i/N}d\eta'_{i/N}}{\pi\hbar N}f_{i/N}(\xi_{i/N},\eta_{i/N})\exp\Big[\frac{2i}{\hbar N}\sum_{i=1}^{N}\begin{vmatrix} 1 & 1 & 1 \\ \xi'_{i+1/N} & \xi_{i/N} & \xi'_{i/N} \\ \eta'_{i+1/N} & \eta_{i/N} & \eta'_{i/N}\end{vmatrix}\Big] \\ &= \int\mathcal{D}\xi\mathcal{D}\eta\mathcal{D}\xi'\mathcal{D}\eta'\prod_{\tau=0}^{1}f(\tau;\xi,\eta)\exp\frac{2i}{\hbar}\int_0^1 d\tau\Big[\frac{d\xi'}{d\tau}(\eta-\eta')-(\xi-\xi')\frac{d\eta'}{d\tau}\Big], \end{aligned} \tag{57}$$

where real fields ξ,η,ξ',η' have boundary conditions

$$x=\xi(0)=\xi(1)=\xi'(0)=\xi'(1)\ ,\ p=\eta(0)=\eta(1)=\eta'(0)=\eta'(1), \tag{58}$$

and functional measures are defined as follows

$$\mathcal{D}\xi := \lim_{N\to\infty}\prod_{i=1}^{N}\frac{d\xi_{i/N}}{\pi\hbar N},\quad \cdots etc. \tag{59}$$

In eq.(57), we can integrate out ξ',η' by using partial integration and obtain simplified form

$$A_\star(f)=\int\mathcal{D}\xi\mathcal{D}\eta\prod_{\tau=0}^{1}f_\tau(\xi(\tau),\eta(\tau))\exp\frac{i}{\hbar}\int_0^1\eta\frac{\partial\xi}{\partial\tau}d\tau. \tag{60}$$

Here we can change the integration area of τ $(0,1)$ to $(-\infty,\infty)$ by a reparametrization of τ . Thus eq.(60) and the boundary conditions (58) are changed as

$$A_\star(f)=\int\mathcal{D}\xi\mathcal{D}\eta\prod_{\tau=-\infty}^{\infty}f_\tau(\xi(\tau),\eta(\tau))\exp\frac{i}{\hbar}\int_{-\infty}^{\infty}\eta\frac{\partial\xi}{\partial\tau}d\tau. \tag{61}$$

$$x=\xi(\pm\infty)=\xi'(\pm\infty),\ p=\eta(\pm\infty)=\eta'(\pm\infty). \tag{62}$$

Next by using eq.(61), we show that the multiple Moyal star product is included in the path integral form of the Kontsevich star product(27). If we put

$$f_\tau(x,p)=\begin{cases} f(x,p) & (\tau=1) \\ g(x,p) & (\tau=0) \\ 1 & (\tau\neq 0,1)\end{cases}\ , \tag{63}$$

then

$$\begin{aligned} f\star g(x,p) &= A_\star(f) \\ &= \lim_{N\to\infty} \cdots\star 1\star f(x,p)\star 1\star\cdots\star 1\star g(x,p)\star 1\star\cdots \\ &= \int \mathcal{D}\xi\mathcal{D}\eta\; f(\xi(1),\eta(1))g(\xi(0),\eta(0))\exp\frac{i}{\hbar}\int_\gamma d^{-1}\omega_0, \end{aligned} \tag{64}$$

where

$$d^{-1}\omega_0 := (d\xi)\eta = \eta\frac{d\xi}{d\tau}d\tau, \tag{65}$$

and

$$\omega_0 = d(d^{-1}\omega_0) = d\xi\wedge d\eta. \tag{66}$$

Eq.(64) corresponds to eq.(27) on the flat plane.

5 Discussions

In this paper, we have proposed the multiple star product method. It is useful for constructing associative star products from *pseudo-associative* star products. We have shown that a *pseudo-associative* S-star product becomes an associative O-star product by using the multiple star product method. It is explained as follows. Although the *pseudo-associative* products break associativity condition a little, the multiple star product method, which is a set of infinite *pseudo-associative* product, smoothes and overcomes this risk and the associativity condition (37) is satisfied. In consequence, the *pseudo-associative* product turns to an associative product within the frame work of the path integral formalism.

The multiple star product method also has been available to well-known associative products e.g. the Moyal star product. The multiple Moyal star product coincides with the path integral form of the Kontsevich star product on the flat plane. This result implies a justice of the multiple star product method.

A Other Examples

By using eq.(61), we can obtain the transition amplitude in quantum dynamics and the bosonic string generating function. If we put in eq.(61)

$$f_\tau(x,p) = \begin{cases} \psi_I(x) & (\tau = t_I) \\ \bar{\psi_F}(x) & (\tau = t_F) \\ e^{-\frac{i}{\hbar}H(x,p)} & (t_I < \tau < t_F) \\ 1 & (\tau < t_I\ ,\ t_F < \tau) \end{cases}, \tag{67}$$

we obtain

$$A_\star(f) \;=\; \lim_{N\to\infty} \cdots\star 1\star\bar{\psi}_F(x)\star e^{-\frac{i\varepsilon}{\hbar}H(x,p)}\star\cdots\star e^{-\frac{i\varepsilon}{\hbar}H(x,p)}\star\psi_I(x)\star 1\star\cdots$$

$$
\begin{aligned}
&= \int \mathcal{D}\xi\mathcal{D}\eta\ \bar{\psi}_F(\xi(t_F))\psi_I(\xi(t_I))e^{\frac{i}{\hbar}\int_{t_I}^{t_F}\left(\eta\frac{\partial\xi}{\partial\tau}-H(\xi,\eta)\right)d\tau} \\
&= \langle\psi_F,t_F|\psi_I,t_I\rangle, \qquad (68)
\end{aligned}
$$

where integration of $\tau < t_I, t_F < \tau$ vanishes because of $f(\tau;x,p)=1$.

Next, a bosonic string generating function can be derived from infinite dimensional multiple Moyal star products. In (61), we put

$$
f_\tau(x,p) = e^{i\left(k(\tau)x+m(\tau)p\right)}, \qquad (69)
$$

and obtain

$$
\begin{aligned}
A_\star(f) &= \lim_{N\to\infty} e^{i\left(k(\infty)x+m(\infty)p\right)}\star\cdots\star e^{i\left(k(0)x+m(0)p\right)}\star\cdots\star e^{i\left(k(-\infty)x+m(-\infty)p\right)} \\
&= \int \mathcal{D}\xi\mathcal{D}\eta\ e^{\frac{i}{\hbar}\int_{-\infty}^{\infty}\left(\eta\frac{\partial\xi}{\partial\tau}+\hbar(k\xi+m\eta)\right)d\tau}. \qquad (70)
\end{aligned}
$$

Here, we generalize $x \to x_{n,\mu}$, $p \to p_n^\mu$ where n runs from 0 to ∞ and μ runs from 1 to dimension d . n can be changed to continuous parameter σ by the following definitions and relations

$$
x_\mu(\sigma) := \sum_{n=0}^{\infty} x_{n,\mu}\cos n\sigma\ ,\ p^\mu(\sigma) := \sum_{n=0}^{\infty}\frac{p_n^\mu}{n}\sin n\sigma,
$$

$$
\sum_{n=0}^{\infty} x_{n,\mu}p_n^\mu = 2\int_0^{2\pi} x_\mu\frac{\partial p^\mu}{\partial\sigma}d\sigma.
$$

Substituting above relation into A_{mn} , we obtain

$$
A_\star(f) = \int \mathcal{D}X \exp\left[\frac{2i}{\hbar}\int_{-\infty}^{\infty}d\tau\int_0^{2\pi}d\sigma\left(\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X_\mu}{\partial\tau}+J^\mu X_\mu\right)\right], \qquad (71)
$$

$$
X_\mu = \frac{\xi_\mu+\eta_\mu}{\sqrt{2}}\ ,\ J^\mu = \hbar\Big(\frac{\partial n^\mu}{\partial\sigma}-\frac{\partial m^\mu}{\partial\sigma}\Big)\ ,\ \mathcal{D}X = \mathcal{D}\xi\mathcal{D}\eta.
$$

$A_\star(f)$ becomes the bosonic string generating function. More details are in [SW].

Acknowledgments

I'm indebted to Professors Y.Maeda and S.Saito for useful discussions.

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